

Jackson's type estimate of nearly coconvex approximation

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Abstract

Suppose that a continuous on the real axis 2π -periodic function f changes its convexity at $2s$, $s \in \mathbb{N}$, points y_i on each period: $-\pi \leq y_{2s} < y_{2s-1} < \dots < y_1 < \pi$, and for the rest $i \in \mathbb{Z}$, the points y_i are defined periodically. In the paper, for each $n \geq N$, a trigonometric polynomial P_n of order cn is found such that: P_n has the same convexity as f , everywhere except, perhaps, the small neighborhoods of the y_i :

$$(y_i - \pi/n, y_i + \pi/n)$$

and

$$\|f - P_n\| \leq c(s) \omega_4(f, \pi/n),$$

where N is a constant depending only on $\min_{i=1, \dots, 2s} \{y_i - y_{i+1}\}$, c and $c(s)$ are constants depending only on s , $\omega_4(f, \cdot)$ is the modulus of continuity of the 4-th order of the function f , and $\|\cdot\|$ is the max-norm.

1 Introduction and the main theorem

By C we denote the space of continuous 2π -periodic functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the uniform norm

$$\|f\| = \max_{x \in \mathbb{R}} |f(x)|,$$

and by \mathbb{T}_n , $n \in \mathbb{N}$, denote the space of trigonometric polynomials

$$P_n(x) = a_0 + \sum_{j=1}^n (a_j \cos jx + b_j \sin jx), \quad a_j \in \mathbb{R}, \quad b_j \in \mathbb{R},$$

of degree $\leq n$. Recall the classical Jackson-Zygmund-Akhiezer-Steckin estimate (obtained by Jackson for $k = 1$, Zygmund and Akhiezer for $k = 2$, and Steckin for $k \geq 3$, $k \in \mathbb{N}$): *if a function $f \in C$, then for each $n \in \mathbb{N}$ there is a polynomial $P_n \in \mathbb{T}_n$ such that*

$$(1.1) \quad \|f - P_n\| \leq c(k) \omega_k(f, \pi/n),$$

where $c(k)$ is a constant depending only on k , and $\omega_k(f, \cdot)$ is the modulus of continuity of order k of the function f . For details, see, for example, [2].

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In 1968 Lorentz and Zeller [8] for $k = 1$ obtained a bell-shaped analogue of the inequality (1.1), *i.e.*, when bell-shaped (even and nonincreasing on $[0, \pi]$) 2π -periodic functions are approximated by bell-shaped polynomials.

In papers [10] and [14] two coconvex analogues of the inequality (1.1) were proved for $k = 2$ and $k = 3$, respectively. Moreover, in [15] arguments from the papers [11], [12] of Shvedov and [1] of DeVore, Leviatan and Shevchuk were used to show that for $k > 3$ there is no coconvex analogue of the inequality (1.1).

Nevertheless, as we know from the coconvex approximation on a closed interval (by algebraic polynomials, see, for details [5]) *if some relaxation of the condition of coconvexity for the approximating polynomials is allowed, then an extra order of the approximation can be achieved*, and, as it seems, no more than one extra order, though the corresponding counterexample is not constructed yet.

So, in the paper in Theorem 1 we prove a trigonometric analogue of the algebraic result [5]. To write it we give necessary notations.

Suppose that on $[-\pi, \pi)$ there are $2s$, $s \in \mathbb{N}$, fixed points y_i :

$$-\pi \leq y_{2s} < y_{2s-1} < \dots < y_1 < \pi,$$

while for other indices $i \in \mathbb{Z}$, the points y_i are defined periodically by the equality

$$y_i = y_{i+2s} + 2\pi \quad (\text{i.e., } y_0 = y_{2s} + 2\pi, \dots, y_{2s+1} = y_1 - 2\pi, \dots).$$

Denote $Y := \{y_i\}_{i \in \mathbb{Z}}$. By $\Delta^{(2)}(Y)$ we denote the set of all functions $f \in C$ which are convex on $[y_1, y_0]$, concave on $[y_2, y_1]$, convex on $[y_3, y_2]$, and so on. The functions in $\Delta^{(2)}(Y_s)$ are *coconvex* with one another. Note, if a function f is twice differentiable, then $f \in \Delta^{(2)}(Y)$ if and only if

$$f''(x)\Pi(x) \geq 0, \quad x \in \mathbb{R},$$

where

$$\Pi(x) := \Pi(x, Y) := \prod_{i=1}^{2s} \sin \frac{x - y_i}{2} \quad (\Pi(x) > 0, \quad x \in (y_1, y_0)).$$

Theorem 1 *If a function $f \in \Delta^{(2)}(Y)$, then there exists a constant $N(Y)$ depending only on $\min_{i=1, \dots, 2s} \{y_i - y_{i+1}\}$ such that for each $n \geq N(Y)$ there is a polynomial $P_n \in \mathbb{T}_{cn}$ satisfying*

$$(1.2) \quad P_n''(x)\Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} (y_i - \pi/n, y_i + \pi/n),$$

$$(1.3) \quad \|f - P_n\| \leq c(s) \omega_4(f, \pi/n),$$

where c and $c(s)$ are constants depending only on s .

The following Theorem 2 is a simple corollary of Theorem 1 and Whitney's inequality [13]

$$\|f - f(0)\| \leq 3 \omega_4(f, 4\pi).$$

Theorem 2 *If a function $f \in \Delta^{(2)}(Y)$, then for each $n \in \mathbb{N}$ there is a polynomial $P_n \in \mathbb{T}_n$ such that*

$$(1.4) \quad P'_n(x)\Pi(x) \geq 0, \quad x \in \mathbb{R} \setminus \cup_{i \in \mathbb{Z}} (y_i - c/n, y_i + c/n),$$

$$(1.5) \quad \|f - P_n\| \leq C(Y) \omega_4(f, \pi/n),$$

where c is a constant depending only on s , and $C(Y)$ is a constant depending only on $\min_{i=1, \dots, 2s} \{y_i - y_{i+1}\}$.

Remark 1 *We believe that ω_4 in (1.3) and (1.5) cannot be replaced by ω_k with $k > 4$. Also we believe that the constants $N(Y)$ and $C(Y)$ in Theorems 1 and 2 cannot be replaced by constants independent of $\min_{i=1, \dots, 2s} \{y_i - y_{i+1}\}$ (and depending, say, on s). These both assumptions are not made further in the paper. Also, we do not pay attention to the constant c in the both theorems, i.e., we did not try to replace it by an absolute constant or/and by a smallest possible one.*

2 Auxiliary facts I

For each $n \in \mathbb{N}$ denote

$$h := h_n := \frac{\pi}{n}, \quad x_j := x_{j,n} := -j h, \quad I_j := I_{j,n} := [x_j, x_{j-1}], \quad j \in \mathbb{Z}.$$

Let $m \in \{1, 2, 3, 10, 20, 30\}$. For a fixed $Y = \{y_i\}_{i \in \mathbb{Z}}$ and a fixed n denote

$$O_{i,m} := O_i(Y, n, m) := (x_{j+m+1}, x_{j-m}) \quad \text{if} \quad y_i \in [x_j, x_{j-1}) =: [x_{j_i}, x_{j_i-1}).$$

Set

$$O_m := O(Y, n, m) := \bigcup_{i \in \mathbb{Z}} O_{i,m}.$$

We will write

$$j \in H(Y, n, m) \quad \text{if} \quad x_j \subset \mathbb{R} \setminus O_m.$$

Let

$$H_m := \{j : j \in H(Y, n, m), |j| \leq n\}.$$

Choose $N(Y) := N(Y, 30) \in \mathbb{N}$ sufficiently large so that

$$(2.1) \quad O_{i,30} \cap O_{i-1,30} = \emptyset$$

for all $n \geq N(Y)$ and all $i = 1, \dots, 2s$ (thus, $N(Y)$ depends only on $\min_{i=1, \dots, 2s} \{y_i - y_{i+1}\}$).

In what follows $n > N(Y)$.

Denote

$$\chi(x, a) := \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a, \end{cases} \quad a \in \mathbb{R}, \quad \chi_j(x) := \chi(x, x_j), \quad (x - x_j)_+ := (x - x_j)\chi_j(x),$$

$$\Gamma_j(x) := \Gamma_{j,n}(x) := \min \left\{ 1, \frac{1}{n \left| \sin \frac{x - (x_j + h/2)}{2} \right|} \right\}, \quad j \in \mathbb{Z}, \quad n \in \mathbb{N},$$

and note that

$$(2.2) \quad \left\| \sum_{j=1-n}^n \Gamma_j^2 \right\| < 6,$$

for details, see [9].

For each $j \in \mathbb{Z}$ and $b \in \mathbb{N}$ we set the positive polynomial $J_j \in \mathbb{T}_{(n-1)b}$, $n \in \mathbb{N}$,

$$(2.3) \quad J_j(x) := J_{j,n}(x) := \left(\frac{\sin \frac{n(x-x_j)}{2}}{\sin \frac{x-x_j}{2}} \right)^{2b} + \left(\frac{\sin \frac{n(x-x_{j-1})}{2}}{\sin \frac{x-x_{j-1}}{2}} \right)^{2b}$$

(i.e., the sum of two "adjacent" kernels of Jackson type).

For each $j \in H_{10}$ denote

$$(2.4) \quad t_j(x) := t_{j,n}(x, b, Y) := \frac{\int_{x_j-\pi}^x J_j(u) \Pi(u) du}{\int_{x_j-\pi}^{x_j+\pi} J_j(u) \Pi(u) du}.$$

In what follows $c_i = c_i(b) = c_i(s, b)$, $i = 1, \dots, 8$, stand for positive constants which may depend only on s and b .

Lemma 3 [7, Lemma 1]. *If $j \in H_{10}$ and $b \geq s + 2$, then*

$$(2.5) \quad t'_j(x) \Pi(x) \Pi(x_j) \geq 0, \quad x \in \mathbb{R},$$

$$(2.6) \quad |\chi_j(x) - t_j(x)| \leq c_1 (\Gamma_j(x))^{2b-2s-1}, \quad x \in [x_j - \pi, x_j + \pi],$$

$$(2.7) \quad |t'_j(x)| \leq c_2 \frac{1}{h} (\Gamma_j(x))^{2b-2s}, \quad x \in \mathbb{R},$$

$$(2.8) \quad |t'_j(x)| \geq c_3 \frac{1}{h} (\Gamma_j(x))^{2b+2s}, \quad x \in \mathbb{R} \setminus O_{10},$$

$$(2.9) \quad |t'_j(x)| \geq c_3 \frac{1}{h} (\Gamma_j(x))^{2b+2s} \left| \frac{x - y_i}{x_j - y_i} \right|, \quad x \in O_{i,10}, \quad i \in \mathbb{Z}.$$

Note that Lemma 3 is proved by using the inequalities

$$(2.10) \quad \begin{aligned} \frac{1}{c_4 h} \Gamma_j^{2b}(x) \left| \frac{\Pi(x)}{\Pi(x_j)} \right| &\leq |t'_j(x)| \leq \frac{c_4}{h} \Gamma_j^{2b}(x) \left| \frac{\Pi(x)}{\Pi(x_j)} \right|, \\ \left| \frac{\Pi(x)}{\Pi(x_j)} \right| &\leq 2^{2s} \Gamma_j^{-2s}(x), \quad x \in \mathbb{R}, \quad j \in H_m, \quad m \geq 10, \end{aligned}$$

$$(2.11) \quad \begin{aligned} \left| \int_{x_j-\pi}^{x_j+\pi} \Gamma_j^b(u) du \right| &\leq c_5 h \Gamma_j^{b-1}(x), \quad b \in \mathbb{N}, \quad x \in [x_j, x_j + 2\pi], \\ \left| \int_x^{x_j-\pi} \Gamma_j^b(u) du \right| &\leq c_5 h \Gamma_j^{b-1}(x), \quad b \in \mathbb{N}, \quad x \in [x_j - 2\pi, x_j], \end{aligned}$$

for details, see [9].

For each $j \in H_{20}$ set the function

$$(2.12) \quad \tau_j(x) := \tau_{j,n}(x, b, t_j) := \alpha \int_{x_j - \pi}^x t_{j+10}(u) du + (1 - \alpha) \int_{x_j - \pi}^x t_{j-10}(u) du,$$

where the number $\alpha \in [0, 1]$ is chosen from the condition

$$\tau_j(x_j + \pi) = \pi$$

(note that the inequalities $0 \leq \alpha \leq 1$ follow from the estimate (2.6) and the choice of the indices $j \pm 10$ if $b \geq s + 2$, for details, see [10, p. 923]).

Note that the functions t_j and τ_j can be expressed on \mathbb{R} as

$$(2.13) \quad t_j(x) = \frac{1}{2\pi}x + \hat{R}_j(x), \quad j \in H_{10},$$

$$(2.14) \quad \tau_j(x) = \frac{1}{4\pi}x^2 + \frac{\pi - x_j}{2\pi}x + \tilde{R}_j(x), \quad j \in H_{20},$$

where \hat{R}_j and \tilde{R}_j are polynomials from \mathbb{T}_{c_6n} (see similar cases in [9] and [10], respectively).

In what follows $c > 0$ denote different absolute constants or constants depending only on s . They can be different even if they are in the same line.

Denote two functions \tilde{t}_j and $\tilde{\tau}_j$:

$$\tilde{t}_j(x) := \tilde{t}_{j,n}(x, b) := \bar{t}_j(x) + \sum_{i=1}^{2s} \frac{\chi_j(y_i) - \bar{t}_j(y_i)}{\hat{t}_{j_i}(y_i)} \hat{t}_{j_i}(x), \quad j \in H_{10},$$

where $\bar{t}_j(x) := t_{j,n}(x, \bar{b}, \emptyset)$ is the function defined by (2.4) with $\Pi(x) \equiv 1$ and $\bar{b} = b + 3$, and

$$\hat{t}_{j_i}(x) := (\bar{t}_{j_i+10}(x) - \check{t}_{j_i-10}(x)) \frac{\Pi(x, Y_i)}{\Pi(x_{j_i}, Y_i)}$$

is the polynomial, where j_i is an index j such that $y_i \in [x_j, x_{j-1})$, $i = 1, \dots, 2s$, $\check{t}_j(x) := t_{j,n}(x, \bar{b}, \check{Y}_i)$ is the function (2.4) with $\check{Y}_i := \{y_i - \pi\nu\}_{\nu \in \mathbb{Z}}$, and

$$Y_i := (Y \setminus \{y_i + 2\pi\nu\}_{\nu \in \mathbb{Z}}) \cup \{y_i^* + 2\pi\nu\}_{\nu \in \mathbb{Z}},$$

where y_i^* is the left endpoint of the interval $O_{i,20}$, if i is odd, and $-$ the right one, if i is even; and

$$\tilde{\tau}_j(x) := \tilde{\tau}_{j,n}(x, b) := \tau_{j,n}(x, b, \bar{t}_j) + \sum_{i=1}^{2s} \frac{(y_i - x_j)_+ - \tau_{j,n}(y_i, b, \bar{t}_j)}{\hat{t}_{j_i}(y_i)} \hat{t}_{j_i}(x), \quad j \in H_{20}.$$

Note that the following Lemma 4 can be proved with the arguments similar to [6, Lemma 5.3].

Lemma 4 [3, Lemmas 4 and 5]. *For each $j \in H_{10}$ and $b \geq 3s + 2$ the function \tilde{t}_j satisfies the relations (2.6), (2.13), and in addition,*

$$(2.15) \quad \left| \chi_j(x) - \tilde{t}_j(x) \right| \leq c_7 (\Gamma_j(x))^{2b-2s-1} \left| \frac{x - y_i}{x_j - y_i} \right|, \quad x \in O_{i,10}, \quad i = 1, \dots, 2s,$$

(in particular, $\chi_j(y_i) - \tilde{t}_j(y_i) = 0$). For each $j \in H_{20}$ and $b \geq 3s + 2$ the function $\tilde{\tau}_j$ satisfies the relation (2.14), and in addition,

$$(2.16) \quad |(x - x_j)_+ - \tilde{\tau}_j(x)| \leq c_8 h (\Gamma_j(x))^{2(b-s-1)}, \quad x \in [x_j - \pi, x_j + \pi],$$

$$(2.17) \quad |(x - x_j)_+ - \tilde{\tau}_j(x)| \leq c_8 h (\Gamma_j(x))^{2(b-s-1)} \left| \frac{x - y_i}{x_j - y_i} \right|, \quad x \in O_{i,10}, \quad i = 1, \dots, 2s,$$

(in particular, $(y_i - x_j)_+ - \tilde{\tau}_j(y_i) = 0$).

3 Auxiliary facts II

Since we prove Theorem 1 using an intermediate approximation by a spline, i.e., the inequality $\|f - S + S - P_n\| \leq \|f - S\| + \|S - P_n\|$, we describe the S in this section. Without special references we will use Whitney inequality [13]

$$|f(x) - L_3(x; a; f)| \leq \omega_4(f, (b-a)/4, [a, b]), \quad x \in [a, b],$$

where L_3 is Lagrange polynomial interpolating f at a , $a + \frac{b-a}{3}$, $b - \frac{b-a}{3}$ and b . Fix $j \in \mathbb{Z}$. Let

$$\Psi_3(x, x_j) := (x - x_j)_+(x - x_{j-1})(x - x_{j-2}), \quad d_j := x_{j-1},$$

$$a_\nu := a_{j,\nu} := x_j \vee x_{j-1} \vee x_{j-2}, \quad \tilde{h}_\nu := -h \vee 0 \vee h, \quad \hat{h}_\nu := 2h^2 \vee -h^2 \vee 2h^2,$$

if $\nu = 1 \vee 2 \vee 3$ respectively. In the following $\nu \in \{1, 2, 3\}$ only.

Introduce three functions $\Psi_{j,\nu} \in \mathbb{C}$ coinciding with $\Psi_3(x, x_j)$ almost everywhere

$$\Psi_{j,\nu}(x) := \Psi_3(x, x_j) \chi(x, a_\nu) = (x - a_\nu)_+^3 + 3\tilde{h}_\nu(x - a_\nu)_+^2 + \hat{h}_\nu(x - a_\nu)_+.$$

That is,

$$(3.1) \quad \Psi_{j,\nu}(x) = \Psi_3(x, x_j), \quad x \in \mathbb{R} \setminus [x_j, a_\nu]; \quad |\Psi_3(x, x_j) - \Psi_{j,\nu}(x)| \leq c h^3, \quad x \in [x_j, a_\nu],$$

$$(3.2) \quad \Psi_{j,\nu}(x) = \int_{d_j - \pi}^x \left(6 \int_{d_j - \pi}^t \left((u - a_\nu)_+ + \tilde{h}_\nu \chi(u, a_\nu) \right) du + \hat{h}_\nu \chi(t, a_\nu) \right) dt,$$

and for $\nu_1, \nu_2 \in \{1, 2, 3\}$, we have

$$(3.3) \quad \frac{\Psi_{j,\nu_1}''(x) - \Psi_{j-1,\nu_2}''(x)}{3h} = \frac{6(x - d_j) - 6(x - d_{j-1})}{3h} = 2, \quad x \in (\max\{a_{\nu_1}, a_{\nu_2}\}, \infty).$$

Without loss of generality suppose that $y_1 = x_{30}$ (i.e., points Y are far from $-\pi$ and π), also recall that $H_3 \subset H_2 \subset H_1$.

Construction of the nearly coconvex cubic spline

Denote two divided differences of f

$$\begin{aligned} F_j &:= [x_j, x_{j-1}, x_{j-2}; f], & j &= 2 - n, \dots, n, \\ \Phi_j &:= [x_{j+1}, x_j, x_{j-1}, x_{j-2}, x_{j-3}; f], & j &= 3 - n, \dots, n - 1. \end{aligned}$$

Remark, $\Phi_j 4h = \frac{F_{j+1}-F_j}{3h} - \frac{F_j-F_{j-1}}{3h}$.

Introduce new functions $\Psi_j(x)$, $j = 3 - n, \dots, n - 1$. For each $j \in H_2$, put

$$(d.0) \quad \Psi_j(x) := \Psi_{j,2}(x) \quad \text{if} \quad \Phi_j \Pi(x_j) \leq 0,$$

otherwise put

$$(d.1) \quad \Psi_j(x) := \begin{cases} \Psi_{j,1}(x) & \text{if } |F_{j+1}| > |F_j| \geq |F_{j-1}|, \\ \Psi_{j,3}(x) & \text{if } |F_{j+1}| \leq |F_j| < |F_{j-1}|, \\ \alpha_j \Psi_{j,1}(x) + (1 - \alpha_j) \Psi_{j,3}(x) & \text{if } |F_{j+1}| > |F_j| < |F_{j-1}|, \end{cases}$$

where $\alpha_j := \frac{F_{j+1}}{F_{j+1}+F_{j-1}} \in (0, 1)$. For other $j = 3 - n, \dots, n - 1$, such that $j \notin H_2$ (i.e., $j : x_j \in O_{i,2}$, $i = 1, \dots, 2s$), put

$$(d.4) \quad \Psi_j(x) := \begin{cases} \Psi_{j,2}(x) & \text{if } \Phi_j \Pi(x_j, \tilde{Y}_i) \leq 0, \\ \Psi_{j,1}(x) & \text{otherwise,} \end{cases} \quad \tilde{Y}_i := (Y \setminus \{y_i\}) \cup \{x_{j_i+5}\}.$$

Set

$$(d.5) \quad \Psi_n(x) := \Psi_3(x, x_n) \quad \Psi_{2-n}(x) \equiv 0.$$

Remark 2 For the both "strange" cases in (d.4) it is sufficient to take simply $\Psi_j(x) = \Psi_{j,2}(x)$ to have the nearly coconvexity of the spline below with f however the setting (d.4) is more convenient to verify the nearly copositivity of P''_n feather.

Show that the cubic spline

$$(3.4) \quad S(x) = L_3(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi_j(x),$$

or equivalently,

$$(3.5) \quad \begin{aligned} S(x) = L_1(x; x_n; f) + F_n \left((x - x_n)(x - x_{n-1}) - \frac{\Psi_n(x) - \Psi_{n-1}(x)}{3h} \right) \\ + \sum_{j=3-n}^{n-1} F_j A_j(x) + F_2 \frac{\Psi_3(x)}{3h}, \end{aligned}$$

where

$$A_j(x) := \overline{A}_j(x) - \underline{A}_j(x) := \frac{\Psi_{j+1}(x) - \Psi_j(x)}{3h} - \frac{\Psi_j(x) - \Psi_{j-1}(x)}{3h},$$

(having been continued periodically) is nearly coconvex with f , i.e.,

$$(3.6) \quad S''(x) \Pi(x) \geq 0, \quad x \in I_j, \quad j \in H_3,$$

and satisfies the inequality

$$(3.7) \quad \|f - S\| = \|f - S\|_{[-\pi, \pi]} \leq c \omega_4(f, h)$$

(it is convenient to look at the sums in (3.4) and (3.5) starting from the last addend, for the details of such a kind of representations, see [4, Proposition 1]).

With the help of (3.4) and (3.5) verify (3.6). Represent the set $[-\pi, \pi] \cap (\bigcup_{j \in H_3} I_j)$, as a union of nonintersecting intervals $[a_\mu, b_\mu]$, $\mu = 1, \dots, 2s+1$, $b_{\mu+1} < a_\mu$. Let $\underline{j} = \underline{j}(\mu)$ and $\bar{j} = \bar{j}(\mu)$ denote the indexes j such that $x_{\underline{j}} = a_\mu$ and $x_{\bar{j}} = b_\mu$, respectively. For each $\mu = 1, \dots, 2s+1$, set

$$G_\mu := \left(d_{\underline{j}+1}, d_{\bar{j}} \right], \quad G := \bigcup_{\mu=1}^{2s+1} G_\mu.$$

Without loose of any generality verify (3.6) only for one interval G_μ , i.e., fix μ , and let it be such that $\Pi(x) > 0$, $x \in G_\mu$. For a conveniens let $n > \underline{j}$ and $\bar{j} > 3 - n$, the cases $n = \underline{j}$ and $\bar{j} = 3 - n$ are proved analogously with respecting (d.5).

Let

$$\overline{H}_\mu := \{ \underline{j} + 1, \dots, \bar{j} \}.$$

Note, $\overline{H}_\mu \subset H_3$. It follows from (3.4), (d.0)-(d.3) that the function S' , at the points a_ν defined separately for each Ψ_j with $j \in \overline{H}_\mu$, satisfies the inequality

$$(3.8) \quad S'(a_\nu -) \leq S'(a_\nu +).$$

Note, $F_j \geq 0$ for $j \in \{ \underline{j} + 2, \dots, \bar{j} - 1 \} =: \overline{\overline{H}}_\mu \subset H_1$. Therefore, in particular, it follows from the inequalities $F_{j+1} \leq F_j \geq F_{j-1}$ that

$$(3.9) \quad \Phi_j \Pi(x_j) \leq 0, \quad j \in \overline{H}_\mu.$$

Taking this into account, remark that in \overline{H}_μ there is not any j for which, in according with the definitions (d.0)-(d.4), the following settings where made

$$\Psi_j = \Psi_{j,3} \quad \text{and} \quad \Psi_{j-1} = \Psi_{j-1,1},$$

as well as the settings like

$$\Psi_{j+1} = \Psi_{j+1,3} \quad \text{and} \quad \Psi_j = \alpha_j \Psi_{j,1} + (1 - \alpha_j) \Psi_{j,3} \quad \text{and} \quad \Psi_{j-1} = \Psi_{j-1,1}.$$

By the other words,

$$(3.10) \quad a_\nu \text{ (defined for } \Psi_j) \leq a_\nu \text{ (defined for } \Psi_{j-1}).$$

From this and (3.3) note,

$$(3.11) \quad A_j''(x) = 0, \quad x \notin (\underline{a}_{j+1}, \overline{a}_{j-1}],$$

where $\underline{a}_j := a_1$ and $\overline{a}_j := a_3$ if (d.3) otherwise $\underline{a}_j = \overline{a}_j$ denote a_ν defined for Ψ_j by (d.0)-(d.2) or (d.4) (if $\underline{a}_{j+1} = \overline{a}_{j-1}$ then $(\underline{a}_{j+1}, \overline{a}_{j-1}] := \emptyset$).

Using the equality $\underline{A}_{j+1} = \overline{A}_j$, extract from (3.5) four addends involving the function Ψ_j

$$(3.12) \quad -F_{j+1} \overline{A}_j(x) + F_j \overline{A}_j(x) - F_j \underline{A}_j(x) + F_{j-1} \underline{A}_j(x).$$

Taking into account (3.9)-(3.12), fix $j \in \overline{H}_\mu$, and show that

$$\begin{aligned} (c.0) & \\ (c.1) & \\ (c.2) & \\ (c.3) & \end{aligned} \quad S''(x) \geq 0, \quad \begin{cases} x \in (a_1, a_3] & \text{if } (d.0), \\ x \in (a_1, a_2] & \text{if } (d.1), \\ x \in (a_2, a_3] & \text{if } (d.2), \\ x \in (a_1, a_3] & \text{if } (d.3), \end{cases}$$

Only these three points a_1 , a_2 and a_3 will take part in the sentences below.

We start from the case (c.1). Describe it on $(a_1, a_2]$. The function Ψ_{j+1} can be stated by (d.0) or (d.4) or (d.1) only, whereas Ψ_{j-1} is any of the four by (d.0)-(d.4). Anyway, $\Psi''_{j+1} = 6(x - a_1)$, $\Psi''_j = 6(x - a_2)$ and $\Psi''_{j-1} = 0$. Hence, by virtue of (17), write

$$F_{j+1} 2 - F_{j+1} 2 + F_j 2 - F_j \frac{6(x - a_2)}{x_{j-3} - x_j} + F_{j-1} \frac{6(x - a_2)}{x_{j-3} - x_j} \geq 0, \quad x \in (a_1, a_2],$$

since $F_j \geq F_{j-1}$.

In the case (c.2) Ψ_{j+1} is any of the four potential settings, whereas Ψ_{j-1} is defined by (d.0) or (d.4) or (d.2) only, but always

$$\overline{A}''_j(x) = 2 + \frac{6(x - a_2)}{x_{j-2} - x_{j+1}} \quad \text{and} \quad \underline{A}''_j(x) = 0 \quad \text{for} \quad x \in (a_2, a_3],$$

where we used (17) in the first equality. Thus,

$$F_{j+1} 2 - F_{j+1} \left(2 + \frac{6(x - a_2)}{x_{j-2} - x_{j+1}} \right) + F_j \left(2 + \frac{6(x - a_2)}{x_{j-2} - x_{j+1}} \right) \geq 0, \quad x \in (a_2, a_3],$$

since $F_{j+1} \leq F_j$.

To see (c.3) note that Ψ_{j+1} and Ψ_{j-1} are defined by (d.0) or (d.4) or (d.1) and by (d.0) or (d.4) or (d.2), respectively. Anyway, $\Psi''_{j+1}(x) = 6(x - a_1)$, $\Psi''_j(x) = \alpha_j 6(x - a_2)$ and $\Psi''_{j-1}(x) = 0$ for $x \in (a_1, a_3]$. Write

$$\begin{aligned} & F_{j+1} 2 - F_{j+1} \frac{6(x - a_1) - \alpha_j 6(x - a_2)}{3h} + F_j \frac{6(x - a_1) - \alpha_j 6(x - a_2)}{3h} \\ & - F_j \frac{\alpha_j 6(x - a_2)}{3h} + F_{j-1} \frac{\alpha_j 6(x - a_2)}{3h} = F_j \left(2 + \frac{(1 - \alpha_j) 6(x - a_2)}{3h} - \frac{\alpha_j 6(x - a_2)}{3h} \right) \\ & + F_{j-1} \frac{\alpha_j 2(x - a_2)}{h} - F_{j+1} \frac{(1 - \alpha_j) 2(x - a_2)}{h} =: B_1(x) + B_2(x). \end{aligned}$$

So, $B_2(x) = 0$ due to the choosing of α_j whereas $B_1(x) \geq 0$ for any $\alpha_j \in [0, 1]$. Really, like (17) rewrite

$$\begin{aligned} B_1(x) &= F_j \left(2 - (1 - \alpha_j) \frac{6(x - a_1) - 6(x - a_2) - 6(x - a_1)}{3h} \right. \\ & \quad \left. - \alpha_j \frac{6(x - a_2) - 6(x - a_3) + 6(x - a_3)}{3h} \right) \\ &= F_j \left(2 - (1 - \alpha_j) 2 + (1 - \alpha_j) \frac{2(x - a_1)}{h} - \alpha_j 2 - \alpha_j \frac{2(x - a_3)}{h} \right) \geq 0, \quad x \in (a_1, a_3]. \end{aligned}$$

For the last case (c.0) note that $\Psi_{j\pm 1}$ can both be any of the four potential settings but it's sufficient to verify this case only when $\Psi_{j+1} = \Psi_{j+1,2}$ and $\Psi_{j-1} = \Psi_{j-1,2}$ because for the other settings the positivity of S'' on $(a_1, a_2] \cup (a_2, a_3]$ is guaranteed by just considered three cases, namely, on $(a_1, a_2]$ - by (c.2) or (c.3), and on $(a_2, a_3]$ - by (c.1) or (c.3). So, for $x \in (a_1, a_3]$ we have

$$\overline{A}''_j(x) = \frac{6(x - a_1) - 6(x - a_2)_+}{3h} \quad \text{and} \quad \underline{A}''_j(x) = \frac{6(x - a_2)_+}{3h},$$

that together with (17) yield

$$F_{j+1} 2 - F_{j+1} \overline{A}_j''(x) + F_j \overline{A}_j''(x) - F_j \underline{A}_j''(x) + F_{j-1} \underline{A}_j''(x) \geq 0.$$

The inequalities (c.0)-(c.3) are proved.

Finally, since the intervals in (c.0)-(c.3) cover all G_μ if j runs through \overline{H}_μ , then

$$(3.13) \quad S''(x) = \sum_{j \in \overline{H}_\mu} F_j A_j''(x) \geq 0, \quad x \in G_\mu,$$

that together with (3.8) leads to (3.6).

To prove (3.7) we need the estimate

$$(3.14) \quad |\Phi_j| \leq c \frac{\omega_4(f, h)}{h^4},$$

see, for example, in [2], (3.1) and the technical spline

$$s(x) = L_3(x; x_n; t) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi_3(x, x_j),$$

that interpolates f without restrictions by cubic parabolas in each x_j , see, [4]. Now let $x \in [x_{j^*+1}, x_{j^*-3}]$, then it follows from (3.4) that

$$\begin{aligned} |f(x) - S(x)| &= |f(x) - s(x) + s(x) + S(x)| \\ &\leq c \omega_4(f, h) + 4h \sum_{j=3-n}^{n-1} |\Phi_j| |\Psi_3(x, x_j) - \Psi_j(x)| \\ &= c \omega_4(f, h) + \sum_{j=\max\{3-n, j^*-3\}}^{\min\{n-1, j^*+3\}} |\Phi_j| 4h |\Psi_3(x, x_j) - \Psi_j(x)| \leq c \omega_4(f, h) \end{aligned}$$

and therefore (3.7) is correct.

4 Proof of Theorem 1

Denote the numbers

$$\begin{aligned} b_1 &:= s + 2, \quad b_2 := 3(s + 1), \\ c_9 &:= \max \left\{ \frac{6((2\pi)^{2b_2} \max\{c_1(b_2), c_7(b_2)\} + c_8(b_2) + 2)}{3c_3(b_1)}, 2 \right\}, \\ n_1 &:= 2[c_9 + 1]n, \quad h_1 := h_{n_1}, \\ c_{10} &:= \max \left\{ c_5(b_2) \left(\frac{c_8(b_2)}{2c_9} + c_1(b_2) \right), 10 \right\}, \\ n_2 &:= 2[c_{10} + 1]n_1, \quad h_2 := h_{n_2}, \end{aligned}$$

where $[\cdot]$ stands for the integer part.

Fix $j = 3-n, \dots, n-1$. For each point a_ν , $\nu = 1, 2, 3$, let j_ν denotes the index such that $x_{j_\nu} := x_{j_\nu, n_1} = a_\nu$, and let j_ν^* denotes the index such that $x_{j_\nu^*} := x_{j_\nu^*, n_2} = x_{j_\nu} (= x_{j_\nu, n_1})$.

Let $j \in H_3$. For each j_ν , $\nu = 1, 2, 3$, we take

$$\tilde{\tau}_{j_\nu}^*(x) = \tilde{\tau}_{j_\nu^*, n_2}^*(x, b_2), \quad \tilde{t}_{j_\nu}^*(x) = \tilde{t}_{j_\nu^*, n_2}^*(x, b_2),$$

and put

$$\begin{aligned} \varphi_{j,\nu}(x) &:= 6 \int_{d_j-\pi}^x \left(\tilde{\tau}_{j_\nu}^*(u) + \tilde{h}_\nu \left(\alpha_\nu \tilde{t}_{(j_\nu+1)^*}(u) + (1 - \alpha_\nu) \tilde{t}_{(j_\nu-1)^*}(u) \right) \right) du, \quad \nu = 1, 3, \\ \varphi_{j,2}(x) &:= 6 \int_{d_j-\pi}^x \left(\tilde{\tau}_{j_2}^*(u) - \frac{1}{12} h^2 \left(\alpha_2 t'_{(j_2+5)^*}(u) + (1 - \alpha_2) t'_{(j_2-5)^*}(u) \right) \right) du, \end{aligned}$$

where $\alpha_\nu \in [0, 1]$, $\nu = 1, 2, 3$, can be chosen such that

$$(4.1) \quad \varphi_{j,\nu}(d_j + \pi) = 3(\pi + h)(\pi - h), \quad \nu = 1, 3, \quad \varphi_{j,2}(d_j + \pi) = 3\pi^2 - \pi h^2/2.$$

Indeed, for example, using (2.16), (2.6) for $\tilde{t}_{j_\nu}^*$, and (2.11), we, for a fixed j , $\nu = 3$ and $\alpha_3 = 1$, have the estimate

$$\begin{aligned} \varphi_{j,3}(d_j + \pi) &= 6 \int_{d_j-\pi}^{d_j+\pi} \left[\tilde{\tau}_{j_3}^*(u) - (u - a_3)_+ + h \left(\tilde{t}_{(j_3+1)^*}(u) - \chi(u, x_{j_3+1}) \right) \right. \\ &\quad \left. + h \left(\chi(u, x_{j_3+1}) - \chi(u, a_3) \right) \right] du + 6 \int_{d_j-\pi}^{d_j+\pi} \left((u - a_3)_+ + h \chi(u, a_3) \right) du \geq 3(\pi^2 - h^2) + 6hh_1 \\ &\quad - 6 \left| \int_{d_j-\pi}^{d_j+\pi} \left[\tilde{\tau}_{j_3}^*(u) - (u - a_3)_+ + h \left(\tilde{t}_{(j_3+1)^*}(u) - \chi(u, x_{j_3+1}) \right) \right] du \right| \geq 3(\pi^2 - h^2) \\ &\quad + 6hh_1 - 6c_8(b_2)h_2 \int_{d_j-\pi}^{d_j+\pi} \Gamma_{j_3^*, n_2}^{2(b_2-s-1)}(u) du - 6c_1(b_2)h \int_{d_j-\pi}^{d_j+\pi} \Gamma_{(j_3+1)^*, n_2}^{2b_2-2s-1}(u) du \\ &\geq 3(\pi^2 - h^2) + 6hh_1 - 6c_5(b_2)(c_8(b_2)h_2^2 + c_1(b_2)hh_2) > 3(\pi^2 - h^2), \end{aligned}$$

whereas for $\alpha_3 = 0$ we (again due to $h_1 \gg h_2$) analogously have the opposite inequality $\varphi_{j,3}(d_j + \pi) < 3(\pi^2 - h^2)$. So, (4.1) is proved for $\nu = 3$, and for $\nu = 1, 2$ it can be proved by analogy.

Now, we take

$$t_{j_\nu}^*(x) = t_{j_\nu^*, n_2}(x, b_2, Y), \quad t_{j_\nu}(x) = t_{j_\nu, n_1}(x, b_1, Y),$$

and put

$$\psi_{j,\nu}(x) := \int_{d_j-\pi}^x \left[\varphi_{j,\nu}(u) + \hat{h}_\nu \left(\beta_\nu t_{(j_\nu+1)^*}(u) + t_{j_\nu}(u) + (1 - \beta_\nu) t_{(j_\nu-1)^*}(u) \right) \right] du,$$

where $\hat{h}_\nu := h^2 \vee -h^2/4 \vee h^2$, $\nu = 1, 2, 3$, respectively.

Lemma 5 *If a fixed j belongs to H_2 , then $\beta_\nu \in [0, 1]$, $\nu = 1, 2, 3$, can be chosen such that*

$$(4.2) \quad \psi_{j,\nu}(d_j + \pi) = (\pi + h)\pi(\pi - h),$$

and then three functions $\psi_{j,\nu}$ satisfy the inequalities

$$(4.3) \quad \begin{aligned} (\psi''_{j,\nu}(x) - \Psi''_{j,\nu}(x)) \Pi(x)\Pi(x_j) &\geq 0, \\ (\psi''_{j,2}(x) - \Psi''_{j,2}(x)) \Pi(x)\Pi(x_j) &\leq 0, \end{aligned} \quad \nu = 1, 3, \quad x \in [-\pi, \pi],$$

$$(4.4) \quad |\Psi_{j,\nu}(x) - \psi_{j,\nu}(x)| \leq c h_{j,n}^3 \Gamma_{j,n}^6(x), \quad \nu = 1, 2, 3, \quad x \in [-\pi, \pi].$$

In addition,

$$(4.5) \quad \begin{aligned} \psi_{j,\nu}(x) &= \frac{1}{8\pi}x^4 + \frac{\pi - d_j}{2\pi}x^3 + \frac{5d_j^2 - 6d_j\pi - h^2}{4\pi}x^2 + \frac{(\pi - d_j)(5d_j^2 - 2\pi^2 - h^2)}{2\pi}x \\ &\quad + Q_{j(\nu)}(x), \quad \nu = 1, 2, 3, \end{aligned}$$

where $Q_{j(\nu)} \in \mathbb{T}_{cn_2}$.

Proof. The relations (4.2)–(4.4) can be proved with the arguments similar to proving (4.1), or [14, Lemma 5], or [3, Lemma 6], using the choice of n_1 and n_2 , and the inequalities $\Gamma_{(j\nu\pm 1)^*, n_2}(x) < \Gamma_{j\nu\pm 1, n_1}(x) < 2\pi\Gamma_{j\nu, n_1}(x) < 2\pi\Gamma_{j,n}(x)$, $x \in \mathbb{R}$. We will calculate here the presentation (4.5) only, with $\nu = 1$, for definiteness. By (2.13) and (2.14) write

$$\tilde{t}_{j_1^*}(x) = \frac{1}{2\pi}x + \hat{R}_{j_1^*}(x), \quad \tilde{\tau}_{j_1^*}(x) = \frac{1}{4\pi}x^2 + \frac{\pi - x_j}{2\pi}x + \tilde{R}_{j_1^*}(x),$$

$$\hat{r}_{j_1^*}(x) := \hat{R}_{j_1^*}(x) - \hat{R}_{j_1^*,0}, \quad \tilde{r}_{j_1^*}(x) := \tilde{R}_{j_1^*}(x) - \tilde{R}_{j_1^*,0},$$

where $\hat{R}_{j_1^*,0}$ and $\tilde{R}_{j_1^*,0}$ are free terms of polynomials $\hat{R}_{j_1^*}, \tilde{R}_{j_1^*} \in \mathbb{T}_{cn}$, respectively. Then

$$\begin{aligned} \varphi_{j,1}(x) &= \left(\frac{1}{2\pi}x^3 + \frac{3(\pi - x_j)}{2\pi}x^2 + 6\tilde{R}_{j_1^*,0}x \right) - \left(\dots(d_j - \pi) \right) \\ &\quad - 6h \left(\frac{1}{4\pi}x^2 + \left(\alpha_1 \hat{R}_{(j_1+1)^*,0} + (1 - \alpha_1) \hat{R}_{(j_1-1)^*,0} \right) x \right) + 6h \left(\dots(d_j - \pi) \right) \\ &\quad + 6 \int_{d_j - \pi}^x \left(\tilde{r}_{j_1^*}(u) - h(\alpha_1 \hat{r}_{(j_1+1)^*}(u) + (1 - \alpha_1) \hat{r}_{(j_1-1)^*}(u)) \right) du \\ &= \frac{1}{2\pi}x^3 + \frac{3(\pi - d_j)}{2\pi}x^2 + 6Ax - \left(\frac{1}{2\pi}(d_j - \pi)^3 + \frac{3(\pi - d_j)}{2\pi}(d_j - \pi)^2 + 6A(d_j - \pi) \right) + q_{j_1}(x), \end{aligned}$$

where

$$A := \tilde{R}_{j_1^*,0} - h \left(\alpha \hat{R}_{(j_1+1)^*,0} + (1 - \alpha) \hat{R}_{(j_1-1)^*,0} \right),$$

and $q_{j_1} \in \mathbb{T}_{cn}$ does not have a free term. Taking this and (4.1) we derive the value of A

$$3(\pi^2 - h^2) = \frac{1}{2\pi}((d_j + \pi)^3 - (d_j - \pi)^3) + \frac{3(\pi - d_j)}{2\pi}((d_j + \pi)^2 - (d_j - \pi)^2) + 12\pi A$$

$$\Rightarrow A = \frac{5d_j^2 - 6d_j\pi - 3h^2}{12\pi},$$

and so,

$$\begin{aligned} \varphi_{j,1}(x) = & \frac{1}{2\pi}x^3 + \frac{3(\pi - d_j)}{2\pi}x^2 + \frac{5d_j^2 - 6d_j\pi - 3h^2}{2\pi}x + \frac{(\pi - d_j)(3d_j^2 - 2d_j\pi - 2\pi^2 - 3h^2)}{2\pi} \\ & + q_{j,1}(x). \end{aligned}$$

Having this, (2.13) and (4.2) we get (4.5) analogously. Lemma 5 is proved. \square

Construction of the nearly coconvex polynomial

For each $j = 3 - n, n - 1$ introduce the polynomial $\psi_j(x) \in \mathbb{T}_{cn_1}$. If $j \in H_2$ then set

$$\psi_j(x) := \psi_{j,2}(x) \quad \text{if} \quad \Phi_j \Pi(x_j) \leq 0,$$

otherwise set

$$\psi_j(x) := \begin{cases} \psi_{j,1}(x) & \text{if } |F_{j+1}| > |F_j| \geq |F_{j-1}|, \\ \psi_{j,3}(x) & \text{if } |F_{j+1}| \leq |F_j| < |F_{j-1}|, \\ \alpha_j \psi_{j,1}(x) + (1 - \alpha_j) \psi_{j,3}(x) & \text{if } |F_{j+1}| > |F_j| < |F_{j-1}|. \end{cases}$$

If $j \notin H_2$ (i.e., $j : x_j \in O_{i,2}, i = 1, \dots, 2s$) then let

$$\psi_j(x) := \begin{cases} \psi_{j,2}(x) & \text{if } \Phi_j \Pi(x_j, \tilde{Y}_i) \leq 0, \\ \psi_{j,1}(x) & \text{otherwise.} \end{cases}$$

Now, put

$$(4.6) \quad P_n(x) = L_3(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j \psi_j(x).$$

The fact that P_n is a polynomial from \mathbb{T}_{cn} can be directly verified arithmetically like in [14], or [3], using (4.5), i.e., all the arithmetical terms in (4.5), having been evaluated in the sum (4) together with the corresponding divided differences, including the L_3 , are equal 0.

Verify (1.2). Remark that Lemma 5 will be used in two senses: in an "ordinary" one for $j \in H_2 = H(n, Y, 2)$, and for $j \notin H_2$ in the sense that $j \in H(n, \tilde{Y}_i, 2)$. So, (4.3), (3.2), (3.4), (3.5) and (3.13) imply

$$\begin{aligned} P_n''(x) \Pi(x) &= \left(L_3''(x; x_n; f) + 4h \sum_{j=3-n}^{n-1} \Phi_j (\psi_j''(x) - \Psi_j''(x)) + 4h \sum_{j=3-n}^{n-1} \Phi_j \Psi_j''(x) \right) \Pi(x) \\ &= 4h \sum_{j \in H_2} \frac{1}{\Pi^2(x_j)} \Phi_j \Pi(x_j) (\psi_{j,\nu}''(x) - \Psi_{j,\nu}''(x)) \Pi(x) \Pi(x_j) \\ &\quad + 4h \sum_{i=1}^{2s} \sum_{j: x_j \in O_{i,1}} \frac{1}{\Pi^2(x_j, \tilde{Y}_i)} \Phi_j \Pi(x_j, \tilde{Y}_i) (\psi_{j,2\nu_1}''(x) - \Psi_{j,2\nu_1}''(x)) \Pi(x, Y) \Pi(x_j, \tilde{Y}_i) \end{aligned}$$

$$+ \left(F_n \left(2 - \frac{\Psi_n''(x) - \Psi_{n-1}''(x)}{3h} \right) + \sum_{j=3-n}^{n-1} F_j A_j''(x) + F_2 \frac{\Psi_3''(x)}{3h} \right) \Pi(x) =: A(x) + B(x) + C(x),$$

$$\begin{aligned} A(x) &\geq 0, & x &\in \mathbb{R}, \\ B(x) &\geq 0, & x &\in \mathbb{R} \setminus \cup_{i \in \mathbb{Z}} (x_{j_i+5}, y_i), \\ C(x) &\geq 0, & x &\in G \text{ on all periods,} \end{aligned}$$

that leads to (1.2). To prove (1.3) we use (3.7), (3.14), (4.4) and (2.2). Namely,

$$\begin{aligned} \|f - P_n\| &= \|f - S + S - P_n\| = \left\| f - S + \sum_{j=3-n}^{n-1} \Phi_j 4h (\Psi_j(\cdot) - \psi_j(\cdot)) \right\|_{[-\pi, \pi]} \\ &\leq c \omega_4(f, h) + c \left\| \sum_{j=3-n}^{n-1} \omega_4(f, h) \Gamma_j^6(\cdot) \right\|_{[-\pi, \pi]} \leq c \omega_4(f, h). \end{aligned}$$

Theorem 1 is proved. □

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